Minimally Incomplete Sampling and Convergence of Adaptive Play in 2×2 Games

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Abstract

Adaptive learning explains how conventions emerge in populations in which players sample a sufficiently small portion of the recent plays and best reply to those samples. We establish that in 2×2 coordination games *any* degree of incomplete sampling is sufficient for a convention to be established and that the degree of sampling does not affect which conventions are most likely to emerge in the long run. Thus, the bound that players sample at most half of the plays available to them, which is prevalent in the large body of work that uses adaptive learning to examine which conventions emerge in a variety of games, is unnecessarily strict.

Keywords: Adaptive play, Incomplete sampling, Conventions, Stochastic stability.

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1 Introduction

With over 1000 citations, Young [1993] is a seminal paper in the field of evolutionary game theory. It introduced a model of learning called adaptive play in which players best respond to a sampled history of play, but occasionally make mistakes and play an action that is not a best response. The paper established that in the unperturbed process (i.e., in the absence of mistakes), play will eventually converge to a convention – a self-enforcing pattern of play in which the same Nash equilibrium is played in each period – if the sampling in the available history by the players is sufficiently incomplete. Through this backward looking best response behavior, Young [1993] offers an explanation for how order and norms can spontaneously evolve in populations. In the perturbed process, it is possible for play to escape a convention and transition to another one. The number of mistakes that are necessary to move the process from one convention into the basin of attraction of another convention is the resistance of moving from the former convention to the latter one, and the stochastically stable conventions – those that require the most mistakes to move from and/or the fewest to move to – are the most likely to be played in the long run.

The question of what portion of records in memory can be sampled to ensure that adaptive play converges to a convention, i.e., how little sampling constitutes "sufficiently" incomplete sampling, has remained largely unaddressed in the literature. In his book *Individual Strategy and Social Structure* (Young [1998]), Young expanded upon the foundation he laid in Young [1993], and proved that in 2×2 coordination games adaptive play converges to a convention if players sample at most half of the records in memory. Young [1998] acknowledges "We do not claim the bound on incompleteness $s/m \leq 1/2$ is the best possible," but to our knowledge there has been no attempt to identify exactly what degree of incomplete sampling is "sufficient." Consequently, follow-up work building upon this theory has retained the bound $s/m \leq 1/2$.[1]

In the current paper, we prove that *any* degree of incomplete sampling is sufficient for the unperturbed adaptive play process to converge to an equilibrium in 2×2 coordination games. In addition, we prove that incomplete sampling is unnecessary in all but some of these games, provided that sample sizes (and thus memory sizes) are large enough. We also prove that increasing the sample size beyond $s/m \leq 1/2$ may result in increased resistances between conventions, but the increased resistances do not affect which conventions are stochastically stable when sampling is incomplete ($s < m$).

[1] Most recently, Proposition 6.4 of Wallace and Young [2015] simply states that in n-player coordination games "if s/m is sufficiently small, the [adaptive learning] process converges with probability one to a convention from any initial state."

2 Adaptive play in 2×2 coordination games

2.1 2×2 coordination games

Consider a 2×2 game $G = (N; A_1, A_2; u_1, u_2)$ with player set $N = \{1, 2\}$, actions sets $A_1 = \{a_1, a_2\}$ and $A_2 = \{b_1, b_2\}$, and payoff functions $u_i : A_1 \times A_2 \to \mathbf{R}$ $(i = 1, 2)$. The game G is a coordination game if it has two pure-strategy Nash equilibria on a diagonal, and, without loss of generality, we assume that (a_1, b_1) and (a_2, b_2) are Nash equilibria: $u_1(a_1, b_1) \geq u_1(a_2, b_1)$ and $u_2(a_1, b_1) \geq u_2(a_1, b_2)$ (i.e., (a_1, b_1) is a Nash equilibrium), and $u_1(a_2, b_2) \geq u_1(a_1, b_2)$ and $u_2(a_2, b_2) \geq u_2(a_2, b_1)$ (i.e., (a_2, b_2) is a Nash equilibrium). Throughout this paper we also assume that for player 1 either $u_1(a_1, b_1) > u_1(a_2, b_1)$ or $u_1(a_2, b_2) > u_1(a_1, b_2)$ and for player 2 either $u_2(a_1, b_1) > u_2(a_1, b_2)$ or $u_2(a_2, b_2) > u_2(a_2, b_1)$. These last two conditions are generically satisfied: they rule out the possibility that one of the players has the same payoff from both their actions regardless of the action played by the other player, in which case the only distinction between a player's two actions is, from their own perspective, the names of the actions. In some of our results, we will use the requirement that at least one of the two Nash equilibria is strict: either $u_1(a_1, b_1) > u_1(a_2, b_1)$ and $u_2(a_1, b_1) > u_2(a_1, b_2)$ so that (a_1, b_1) is strict, or $u_1(a_2, b_2) > u_1(a_1, b_2)$ and $u_2(a_2, b_2) > u_2(a_2, b_1)$ so that (a_2, b_2) is strict.

Because each player has only two actions in the game G, every mixed strategy p_i of player $i \in \{1, 2\}$ can be identified by the probability $p_i(s_i)$ with which player i plays one of their actions s_i (because that leaves probability $1 - p_i(s_i)$ that player i plays their other action). Action a_1 is a best response by player 1 to a mixed strategy p_2 of player 2 if and only if $p_2(b_1) \geq \frac{u_1(a_2,b_2)-u_1(a_1,b_2)}{u_1(a_1,b_1)-u_1(a_1,b_2)-u_1(a_2,b_1)+u_1(a_2,b_2)}$. Note that $\alpha_2 := \frac{u_1(a_2,b_2)-u_1(a_1,b_2)}{u_1(a_1,b_1)-u_1(a_1,b_2)-u_1(a_2,b_1)+u_1(a_2,b_2)} \in [0, 1]$ because G is a coordination game with Nash equilibria (a_1, b_1) and (a_2, b_2), and either $u_1(a_1, b_1) > u_1(a_2, b_1)$ or $u_1(a_2, b_2) > u_1(a_1, b_2)$.[2] Similarly, action b_1 is a best response by player 2 to a mixed strategy p_1 of player 1 if and only if $p_1(a_1) \geq \alpha_1 := \frac{u_2(a_2,b_2)-u_2(a_2,b_1)}{u_2(a_1,b_1)-u_2(a_2,b_1)-u_2(a_1,b_2)+u_2(a_2,b_2)} \in [0, 1]$.

2.2 Adaptive play in 2-player games

We study adaptive play [Young, 1993] with memory m and sample size $1 < k \leq m$ of the game G, as explained below.[3]

For each role (player position) $i \in N$ in game G, there is a class of players C_i who can play that role.

[2] The only possible hick-up is that the denominator could equal 0, but that is ruled out when player 1 has two actions that differ to them in more than name only.

[3] Throughout, we use k for the sample size because we already use s for strategies.

No player can play in more than one role ($C_1 \cap C_2 = \emptyset$). In each period t, a player is drawn from each class, and the two players that are drawn play the game G – each player i chooses an action $s_i(t) \in A_i$ from the actions available to them in their role. The action-tuple $s(t) = (s_1(t), s_2(t))$ is recorded and will be referred to as the play at time t. The history of plays up to and including time t is the ordered vector $h(t) = (s(1), s(2), s(3), ..., s(t))$, and the history of the last m plays, called a state, is the ordered vector $h(t|m) = (s(t-m+1), s(t-m+2), ..., s(t))$.

In period $t+1$, the player in role i draws a sample R_i^{t+1} of size k from the m most recent plays $s_j(t-m+1), s_j(t-m+2), \ldots, s_j(t)$ by the players in role $j \neq i$. Player i predicts that the players in role j play a mixed strategy $p_j(\cdot|R_i^{t+1})$ that is the frequency distribution of the actions in the sample drawn: $p_j(s_j|R_i^{t+1})$ equals the number of times that action s_j occurs in the sample R_i^{t+1} divided by k, for each $s_j \in A_j$. Player i then plays an action that is a best response to this predicted mixed strategy: $s_i(t+1) \in BR_i(R_i^{t+1}) := \arg\max \{\sum_{s_j \in A_j} \left(p_j(s_j|R_i^{t+1}) \cdot u_i(s_i, s_j)\right) \mid s_i \in A_i\}$.

The decision making process described above is called unperturbed adaptive play with memory size m and sample size k. Through an adaptive play process, self-enforcing patterns of play, called conventions, can emerge.

Definition 1. *A* convention *is a state $h(t|m)$ that consists of m repetitions of the same Nash equilibrium s^* of the game G.*

When a convention is reached in which the Nash equilibrium s^* is played, then the players can only sample the others playing their part of s^* and thus all players have a best response to play their part of s^*. That means that adaptive play predicts that the players can keep playing s^* in all subsequent periods. If the Nash equilibrium s^* is strict, then the best responses are unique and, without perturbations, the players will keep playing s^* indefinitely.

3 Minimally incomplete sampling

Young [1998] proved that in 2×2 coordination games, unperturbed adaptive play will reach a convention as long as sampling is sufficiently incomplete. Incomplete sampling means that the players sample only a faction of the records in memory and in Young [1998] the specific limit for sampling to be "sufficiently" incomplete is $k \leq \frac{m}{2}$, meaning that players sample at most half of all the records available in memory. We relax this bound substantially and show that in 2×2 coordination games with at least one strict Nash equilibrium, *any* degree of incomplete sampling is sufficient for a convention to eventually be reached, and that in most of these games sample size equal to memory size (i.e., complete sampling) suffices when the memory size is large enough.

Lemma 1 will be used in the proofs of Theorems 1, 2 and 3.

Lemma 1. *Let G be a 2×2 coordination game and let $s^* = (s_1^*, s_2^*)$ be a (pure-strategy) Nash equilibrium of G. Consider unperturbed adaptive play with memory size m and sample size $k \leq m$. Let $t > m$ be a period in which each player $i \in \{1, 2\}$ can play s_i^* as a best response to their sampled history, so that there is a positive probability that the strategy-tuple s^* is played in period t. Then the convention of playing s^* can be reached with positive probability.*

Proof of Lemma 1. Using induction, we show that there exists a positive probability that s^* is played in periods t through $t + m - 1$, so that the convention of playing s^* is reached.

Base Step: By assumption, the strategy-tuple $s^* = (s_1^*, s_2^*)$ is played with positive probability in period t.

Inductive Step: Let $\hat{t} \geq t$ and suppose that it has already been demonstrated that each player $i \in \{1, 2\}$ can play s_i^* as a best response to their sampled history in period $\hat{t}$, so that there is a positive probability that the strategy-tuple s^* is played in period $\hat{t}$. It will be shown that there is a positive probability that s^* is played in period $\hat{t} + 1$ as part of adaptive play.

For each player $i \in \{1, 2\}$, let $R_i^{\hat{t}}$ be a sampled history of player i in period $\hat{t}$ such that $s_i^* \in BR_i(R_i^{\hat{t}})$, and let $s_i(\hat{t}) = s_i^*$. Then there is a positive probability that each player i draws a sample $R_i^{\hat{t}+1}$ that is obtained by replacing one of the records in $R_i^{\hat{t}}$ with $s_j(\hat{t}) = s_j^*$ $(j \neq i)$. If the replaced record is equal to s_j^*, then this does not change the frequency of s_j^* in i's sample, and if the replaced record is not equal to s_j^*, then this increases the frequency of s_j^* in i's sample. If $s^* = (a_1, b_1)$, then $p_j(s_j^* | R_i^{\hat{t}+1}) \geq p_j(s_j^* | R_i^{\hat{t}}) \geq \alpha_j$, where the last step holds because $s_i^* \in BR_i(R_i^{\hat{t}})$. Similarly, if $s^* = (a_2, b_2)$, then $p_j(s_j^* | R_i^{\hat{t}+1}) \geq p_j(s_j^* | R_i^{\hat{t}}) \geq 1 - \alpha_j$. In both cases, it follows that $s_i^* \in BR_i(R_i^{\hat{t}+1})$.

Therefore, there is a positive probability that s^* is played in period $\hat{t} + 1$ as part of adaptive play.

Conclusion: Using the inductive step $m - 1$ times, it has thus been shown that there exists a positive probability that s^* is played in periods t through $t + m - 1$, so that the convention of playing s^* is reached.
■

Lemma 1 exploits the fact that in a 2×2 game, when player i's Nash equilibrium action s_i^* is a best response to the other player j's mixed strategy, and subsequently, the probability that player j plays s_j^* (weakly) increases, then s_i^* is still a best response by player i. Loosely speaking, it seems fairly intuitive that when the other player plays their Nash equilibrium action with larger probability, this will increase a player's incentive to play their best response to that action. However, this intuition does not extend to larger games and the statement of the lemma is not necessarily true for such games.[4]

The result in Lemma 1 allows us to establish that *any* incomplete sampling, even just by one record, is sufficient to guarantee that adaptive play converges to a strict Nash equilibrium in 2×2 coordination games.

Theorem 1. *Let G be a 2×2 coordination game in which at least one of the two Nash equilibria is strict. From any initial state, unperturbed adaptive play with memory size m and sample size $k < m$ converges with probability 1 to a convention corresponding to a strict Nash equilibrium and locks in.*

Proof of Theorem 1. In light of Lemma 1, it suffices to demonstrate that there exists a period $t > m$ in which a strict Nash equilibrium $s^* = (s_1^*, s_2^*)$ is played with positive probability, because then the convention of playing s^* can be reached with positive probability, and once that convention is reached, the players will keep playing s^* indefinitely.

Without loss of generality, assume that the Nash equilibrium (a_1, b_1) is strict. Consider unperturbed adaptive play with memory size m and sample size $k < m$ starting from an arbitrary initial state. Consider an arbitrary period $t > m$ and the history $h(t) = (s(1), s(2), s(3), ..., s(t))$ of plays up to and including time t. We distinguish three cases.

Case 1. In period $t+1$ it is possible for the players to draw samples R_i^{t+1}, $i = 1, 2$, such that $a_1 \in BR_1(R_1^{t+1})$ and $b_1 \in BR_2(R_2^{t+1})$. Then there is a positive probability that $s(t+1) = (a_1, b_1)$.

Case 2. In period $t+1$ it is possible for the players to draw samples R_i^{t+1}, $i = 1, 2$, such that $a_2 \in BR_1(R_1^{t+1})$ and $b_2 \in BR_2(R_2^{t+1})$. There is a positive probability that $s(t+1) = (a_2, b_2)$. If the Nash equilibrium (a_2, b_2) is strict, then we have reached a period in which the players play a strict Nash equilibrium.

If the Nash equilibrium (a_2, b_2) is not strict, then $u_1(a_2, b_2) = u_1(a_1, b_2)$ or $u_2(a_2, b_2) = u_2(a_2, b_1)$ (or both). Assume, without loss of generality, that $u_1(a_2, b_2) = u_1(a_1, b_2)$ (and $u_2(a_2, b_2) \geq u_2(a_2, b_1)$). Then $BR_1(R_1^{t+1}) = \{a_1, a_2\}$ and thus $a_1 \in BR_1(R_1^{t+1})$. Thus, $s(t+1) = (a_1, b_2)$ is played with positive probability in the adaptive play process. For the next $k-1$ periods, regardless of the actions that player 2 plays and the samples that player 1 draws, player 1 can keep playing $s_1(\hat{t}) = a_1$, $\hat{t} = t+2, \ldots, t+k$, as a best response. Then in period $t+k+1$, player 2 can draw a sample R_2^{t+k+1} from player 1's actions that consists of k instances of player 1 playing a_1, so that $b_1 \in BR_2(R_2^{t+k+1})$. Thus, there is a positive probability that $s(t+k+1) = (a_1, b_1)$.

[4]Examples are available form the authors upon request.

Case 3. If in period $t+1$ it is not possible for the players to draw samples R_i^{t+1}, $i = 1, 2$, such that $s_i \in BR_i(R_i^{t+1})$ for $i = 1, 2$ and (s_1, s_2) is a Nash equilibrium of G, then, without loss of generality, assume that $BR_1(R_1^{t+1}) = \{a_1\}$ for all samples that player 1 can draw, and $BR_2(R_2^{t+1}) = \{b_2\}$ for all samples that player 2 can draw, so that $s(t+1) = (a_1, b_2)$.

This implies that in $h(t|m)$ player 2 played b_2 at most β_2 times, where β_2 is the largest number in $\{0, 1, \ldots, k-1\}$ that is strictly lower than $(1-\alpha_2) \times k$.[5] Similarly, in $h(t|m)$ player 1 played a_1 at most β_1 times, where β_1 is the largest number in $\{0, 1, \ldots, k-1\}$ that is strictly lower than $\alpha_1 \times k$.[6] However, $s(t+1) = (a_1, b_2)$, so that the number of times that player 1 (resp. 2) plays action a_1 (resp. b_2) in $h(t+1|m)$ is either equal to that in $h(t|m)$ (in case $s_1(t-m+1) = a_1$, resp. $s_2(t-m+1) = b_2$) or one higher. As long as these numbers do not exceed β_1, resp. β_2, the players will keep playing $s(\hat{t}) = (a_1, b_2)$ in periods $\hat{t} \geq t+2$. This clearly cannot persist because after m periods the players would only have plays (a_1, b_2) in recent memory.

Let $\hat{t} \geq t+1$ be the first period in which either player 1 played a_1 more than β_1 times in $h(\hat{t}|m)$ or player 2 played b_2 more than β_2 times in $h(\hat{t}|m)$ (or both). Without loss of generality, assume that player 1 played β_1 instances of a_1 in $h(\hat{t}-1|m)$ and β_1+1 instances of a_1 in $h(\hat{t}|m)$. Then in period $\hat{t}+1$, player 2 can draw a sample $R_2^{\hat{t}+1}$ that contains β_1+1 instances of player 1 playing a_1, and play $s_2(\hat{t}+1) = b_1 \in BR_2(R_2^{\hat{t}+1})$. Also, player 2 played at most β_2 instances of b_2 in $h(\hat{t}-1|m)$, and thus at most β_2+1 instances of b_2 in $h(\hat{t}|m)$. Thus, because $k < m$, in period $\hat{t}+1$, player 1 can draw a sample $R_1^{\hat{t}+1}$ that contains no more than β_2 instances of player 2 playing b_2, and play $s_1(\hat{t}+1) = a_1 \in BR_1(R_1^{\hat{t}+1})$. Thus, there is a positive probability that $s(\hat{t}+1) = (a_1, b_1)$.

Conclusion. The three cases we considered are exhaustive and thus we have shown that, starting from any period $t > m$ and with any history of play at that time, we can find a period in which there is a positive probability that the players play a strict Nash equilibrium in the adaptive play process with sample size $k < m$. Lemma 1 then establishes that the convention of playing that strict Nash equilibrium can be reached with positive probability, and then the process is locked in. ■

Note that in the proof of Theorem 1, there is only one instance in which we use that sampling is incomplete ($k < m$), and that is in Case 3, where we need it to guarantee that the adaptive play process cannot get "stuck" in a situation where both players mis-coordinate in every period, oscillating between (a_1, b_2) and

[5] We remind the reader that α_2 is the probability such that action a_1 is a best response by player 1 to a mixed strategy p_2 of player 2 if and only if $p_2(b_1) \geq \alpha_2$. Also, because (a_1, b_1) is a strict Nash equilibrium, $\alpha_2 < 1$, so that $(1-\alpha_2) \times k > 0$.

[6] We remind the reader that α_1 is the probability such that action b_1 is a best response by player 2 to a mixed strategy p_1 of player 1 if and only if $p_1(a_1) \geq \alpha_1$. Note that $\alpha_1 > 0$, because otherwise $b_1 \in BR_2(R_2^{t+1})$ regardless of the sample that player 2 draws.

(a_2, b_1) and necessarily switching actions in exactly the same periods. If the game and sample sizes are such that this cannot happen anyway, then we do not need sampling to be incomplete at all, and we can have $k = m$. We use the notation $\lceil\cdot\rceil$ to denote the ceiling function, which rounds up any real number to the smallest natural number that is at least as large.[7]

Theorem 2. *Let G be a 2×2 coordination game in which at least one of the two Nash equilibria is strict and such that $\alpha_1 \neq 1-\alpha_2$. Let the sample size k be such that $\lceil\alpha_1 \times k\rceil \neq \lceil(1-\alpha_2)\times k\rceil$ or $\lceil\alpha_2 \times k\rceil \neq \lceil(1-\alpha_1)\times k\rceil$. From any initial state, unperturbed adaptive play with memory size m and sample size $k \leq m$ converges with probability 1 to a convention corresponding to a strict Nash equilibrium and locks in.*

Proof of Theorem 2. If $k < m$, then Theorem 1 applies. So, suppose that $k = m$, i.e, sampling is complete in the sense that players see *all* of the past m records.

Consider an adaptive play process with $k = m$. If in some period $t > m$ the players coordinate, i.e., $s(t) = (a_1, b_1)$ or $s(t) = (a_2, b_2)$, then we can apply cases 1 or 2 in the proof of Theorem 1 to establish that there is a positive probability that the players play a strict Nash equilibrium (note that these cases do not depend on $k < m$). Lemma 1 then establishes that the convention of playing that strict Nash equilibrium can be reached with positive probability, and then the process is locked in.

Thus, it remains to consider the possibility that the players mis-coordinate in all periods, i.e., $s(t) \in \{(a_1, b_2), (a_2, b_1)\}$ for all t. We will demonstrate that this cannot happen because an implication of the assumption that $\lceil\alpha_1 \times k\rceil \neq \lceil(1-\alpha_2)\times k\rceil$ or $\lceil\alpha_2 \times k\rceil \neq \lceil(1-\alpha_1)\times k\rceil$ is that an adaptive play process with $k = m$ cannot result in string of mis-coordinated plays $s(1), s(2), \ldots$ with $s(t) \in \{(a_1, b_2), (a_2, b_1)\}$ for all t. Without loss of generality assume $\lceil\alpha_1 \times m\rceil \neq \lceil(1-\alpha_2)\times m\rceil$. If, in some period $t > m$, the players observe a history of play that consists of a string of m instances of (a_1, b_2) having been played in the previous m periods, player 2's unique best response is to play b_1 in the next period or player 1's unique best response is to play a_2 in the next period.[8] Thus, any string of mis-coordinated plays that contains a string of more than m subsequent plays of (a_1, b_2) cannot be the result of an adaptive play process. Similarly, any string of mis-coordinated plays that contains a string of more than m subsequent plays of (a_2, b_1) cannot be the result of an adaptive play process. We conclude that if the players mis-coordinate in all periods, and they follow an adaptive play process, then the process needs to switch repeatedly between playing (a_1, b_2) and (a_2, b_1).

For player 1 to switch to playing a_2, they need to observe $\lceil(1-\alpha_2)\times m\rceil$ instances of player 2 playing

[7] So, if n is a natural number itself, then $\lceil n\rceil = n$. Also, we include 0 in the set of natural numbers.

[8] This uses $\lceil\alpha_1 \times m\rceil \neq \lceil(1-\alpha_2)\times m\rceil$, which implies that it cannot be the case that player 1 can best respond by playing a_1 *and* player 2 can best respond by playing b_2 after both observe m instances of (a_1, b_2) having been played.

b_2, and for player 2 to switch to playing b_1, they need to observe $\lceil \alpha_1 \times m \rceil$ instances of player 1 playing a_1. However, in all periods $t > m$, because $k = m$, player 1 samples as many records of player 2 playing b_2 as player 2 samples records of player 1 playing a_1. Thus, $\lceil \alpha_1 \times m \rceil \neq \lceil (1 - \alpha_2) \times m \rceil$ implies that the players will not switch from playing (a_1, b_2) to playing (a_2, b_1) in the same period when they follow an adaptive play process. ■

The interpretation of the condition $\alpha_1 \neq 1 - \alpha_2$ in the statement of Theorem 2 is of course that the smallest probability for player 1 to play a_1 such that action b_1 is a best response by player 2 is not equal to the smallest probability for player 2 to play b_2 such that action a_2 is a best response by player 1. This is a very weak condition that is generically satisfied. If the game G is such that α_1 and $1 - \alpha_2$ are close, then the condition that $\lceil \alpha_1 \times k \rceil \neq \lceil (1 - \alpha_2) \times k \rceil$ or $\lceil \alpha_2 \times k \rceil \neq \lceil (1 - \alpha_1) \times k \rceil$ will require a large sample size.[9]

4 Stochastic stability of conventions

4.1 Perturbed adaptive play and resistances

In perturbed adaptive play, in every round players play a best response to their sample R_i^t with probability $1 - \varepsilon$, and they play a random action with probability ε. Thus, if one of player i's actions is not a best response to their sample, then they can still play it by mistake. Allowing for mistakes makes transitions possible between conventions, even those in which a strict Nash equilibrium is played. Denote by h_i, $i = 1, 2$, the convention corresponding to the Nash equilibrium (a_i, b_i), i.e., the state that consists of m repetitions of (a_i, b_i). Now consider the transition from h_i to h_j, $i \neq j$. The *resistance* $r_{i,j}^{k,m}$ is the minimum number of mistakes necessary to make the transition from h_i to h_j in the perturbed adaptive play process. Young [1998] shows that when at most half of all records in memory can be sampled ($k \leq m/2$), the resistance of moving from h_2 to h_1 equals $\min\big(\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big)$ and the resistance of moving from h_1 to h_2 equals $\min\big(\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\big)$, and the resistances are thus independent of m. However, we demonstrate below that when sampling is less incomplete ($k > m/2$), the resistances may be larger and depend on m.

Theorem 3. *Consider perturbed adaptive play with memory size m and sample size $k \leq m$ in a 2×2 coordination game G. The resistance of moving from h_2 to h_1 equals $r_{2,1}^{k,m} = \min\big\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big\} + \max\big\{\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil - m, 0\big\}$ and the resistance of moving from h_1 to h_2 equals $r_{1,2}^{k,m} = \min\big\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1 - \alpha_2) \times k \rceil\big\} + \max\big\{\lceil (1 - \alpha_1) \times k \rceil + \lceil (1 - \alpha_2) \times k \rceil - m, 0\big\}$.*

[9] Note that $\lceil \alpha_1 \times k \rceil \neq \lceil (1 - \alpha_2) \times k \rceil$ does not necessarily imply $\lceil \alpha_2 \times k \rceil \neq \lceil (1 - \alpha_1) \times k \rceil$. An illustration of this can be found in Example 1.

Proof of Theorem 3. We compute the resistance $r_{2,1}^{k,m}$. Similarly to Case 2 in the proof of Theorem 1, we derive that no mistakes are necessary to move from h_2 to h_1 if equilibrium (a_2, b_2) is not strict. In that case, either $\alpha_1 = 0$ or $\alpha_2 = 0$ or both hold and the expression for $r_{2,1}^{k,m}$ in the statement of the theorem indeed produces a resistance equal to 0.

So assume that equilibrium (a_2, b_2) is strict and let t be a period such that $h(t|m) = h_2$, i.e., the system is in convention h_2. Because (a_2, b_2) is strict, (a_2, b_2) will continue to be played if the players do not make any mistakes. To reach convention h_1, it is necessary to reach a period in which both a_1 and b_1 can be played as best responses to samples drawn by the players.[10] Reaching a period in which both a_1 and b_1 can be played as best responses to samples drawn by the players is also a sufficient condition for the process to reach convention h_1 without further mistakes (see Lemma 1). Thus, starting from convention h_2, we need to determine the minimum number of mistakes (which will be positive) necessary to build a length-m history of play from which both players can draw samples of size k such that a_1 and b_1 are best responses. For this condition to be met in some period T, in periods $T-m$ through $T-1$ player 1 must have played a_1 at least $\lceil \alpha_1 \times k \rceil$ times and player 2 must have played b_1 at least $\lceil \alpha_2 \times k \rceil$ times. Clearly, this can be accomplished by having player 1 make a mistake and play a_1 a total of $\lceil \alpha_1 \times k \rceil$ times *and* having player 2 make a mistake and play b_1 a total of $\lceil \alpha_2 \times k \rceil$ times in $\max\left\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\right\}$ consecutive periods. This gives an upper bound of $\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil$ for $r_{2,1}^{k,m}$.

The number of mistakes can be lowered by decreasing the number of periods in which both players make a mistake, so that players can sample each other's mistakes and potentially play a_1 and/or b_1 as best responses. At the extreme, when sample sizes are sufficiently incomplete so that players can keep sampling mistakes long enough, it suffices for one player to make enough mistakes to make their action in (a_1, b_1) a best response by the other player, and we obtain the lower bound $\min\left\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\right\}$ for $r_{2,1}^{k,m}$. We consider this case first.

Case 1. $\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil \leq m$.[11]

Starting in period $t+1$, suppose player 1 makes $\lceil \alpha_1 \times k \rceil$ consecutive mistakes and plays a_1 in periods $t+1, \ldots, t+\lceil \alpha_1 \times k \rceil$. During each of these periods, player 2 can sample no more than $\lceil \alpha_1 \times k \rceil - 1$ instances of player 1 playing a_1 and can only play b_2 as a best response.

Because $\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil \leq m$, in each of the periods $t+\lceil \alpha_1 \times k \rceil + 1$ through $t+\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil$, player 2 can sample all $\lceil \alpha_1 \times k \rceil$ instances of a_1 that player 1 played in periods $t+1$ to $t+\lceil \alpha_1 \times k \rceil$ and play b_1 as a best response. In these periods, player 1 can sample no more than $\lceil \alpha_2 \times k \rceil - 1$ plays of b_1 and can

[10] Note that this is a property that is satisfied by convention h_1.
[11] In this case, derivations are similar to those in Young [1998].

only play a_2 as a best response.

	Play										
Period	$t-m+1$	$\dots$	t	$t+1$	$\dots$	$t+\lceil\alpha_1\times k\rceil$	$t+\lceil\alpha_1\times k\rceil+1$	$\dots$	$t+\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil$	$t+\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil+1$	$\dots$
Player 1	a_2	a_2	a_2	a_1	a_1	a_1	a_2	a_2	a_2	a_1	a_1
Player 2	b_2	b_2	b_2	b_2	b_2	b_2	b_1	b_1	b_1	b_1	b_1

The color red denotes actions which necessarily are mistakes. Actions colored blue can be played as a best response.

Because $\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil\leq m$, in period $t+\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil+1$ it is possible for player 2 to sample all $\lceil\alpha_1\times k\rceil$ player 1's plays of a_1 in periods $t+1$ through $t+\lceil\alpha_1\times k\rceil$, while player 1 samples all $\lceil\alpha_2\times k\rceil$ player 2's plays of b_1 in periods $t+\lceil\alpha_1\times k\rceil+1$ through $t+\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil$. Thus, both a_1 and b_1 can be played as best responses by the players in period $t+\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil+1$ and the process can reach convention h_1 without further mistakes (see Lemma 1).

The process we just described reaches convention h_1 from convention h_2 with exactly $\lceil\alpha_1\times k\rceil$ mistakes by starting in period $t+1$ with player 1 making $\lceil\alpha_1\times k\rceil$ consecutive mistakes and playing a_1 in periods $t+1,\dots,t+\lceil\alpha_1\times k\rceil$. If instead we start in period $t+1$ with player 2 making $\lceil\alpha_2\times k\rceil$ consecutive mistakes and playing b_1 in periods $t+1,\dots,t+\lceil\alpha_2\times k\rceil$, we obtain a process that reaches convention h_1 from convention h_2 with exactly $\lceil\alpha_2\times k\rceil$ mistakes.

Because either player 1 must have played a_1 at least $\lceil\alpha_1\times k\rceil$ times to allow player 2 to play b_1 as a best response, or player 2 must have played b_1 at least $\lceil\alpha_2\times k\rceil$ times to allow player 1 to play a_1 as a best response, the minimum number of mistakes necessary to reach convention h_1 from convention h_2 equals $\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}$. Since (at least) one of the two processes we described reaches convention h_1 from convention h_2 with exactly $\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}$ mistakes, we have demonstrated that $r_{2,1}^{k,m}=\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}=\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}+\max\{\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil-m,0\}$.[12]

Case 2. $\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil>m$.

In order to make the transition from h_2 to h_1, at least $\lceil\alpha_1\times k\rceil$ plays of b_1 *and* $\lceil\alpha_2\times k\rceil$ plays of a_1 must occur within m periods. However, $\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil>m$ implies that in order to achieve this condition, a_1 and b_1 must be played in the same period a minimum of $\ell:=\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil-m$ times.[13] Because player 1 cannot play a_1 as a best response until player 2 has played b_1 at least $\lceil\alpha_2\times k\rceil$ times, and player 2 cannot play b_1 as a best response until player 1 has played a_1 at least $\lceil\alpha_1\times k\rceil$ times, the ℓ concurrent plays of a_1 and b_1 require $2\times\ell$ mistakes. Thus, in this case, we need at least an additional ℓ mistakes compared to Case 1. We demonstrate that we do not need more than an additional ℓ mistakes by describing a transition from h_2 to h_1 with exactly $\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}+\max\{\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil-m,0\}$ mistakes.

[12] Note that when $k\leq m/2$, the condition $\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil\leq m$ is satisfied regardless of the values of α_1 and α_2. This is why in Young [1998] the resistance $r_{2,1}^{k,m}$ is given as $r_{2,1}^{k}=\min\{\lceil\alpha_1\times k\rceil,\lceil\alpha_2\times k\rceil\}$.

[13] Note that $\ell=\lceil\alpha_1\times k\rceil+\lceil\alpha_2\times k\rceil-m\leq m$.

Starting in period $t+1$, suppose player 1 makes $\lceil \alpha_1 \times k \rceil$ consecutive mistakes and plays a_1 in periods $t+1, \ldots, t+\lceil \alpha_1 \times k \rceil$. During each of these periods, player 2 can sample no more than $\lceil \alpha_1 \times k \rceil - 1$ instances of player 1 playing a_1 and can only play b_2 as a best response. Suppose that in the last ℓ of these periods, $t+m-\lceil \alpha_2 \times k \rceil + 1$ through $t+\lceil \alpha_1 \times k \rceil$, player 2 makes ℓ consecutive mistakes and plays b_1.

	Play													
Period	$t-m+1$	$\ldots$	t	$t+1$	$\ldots$	$t+m-\lceil \alpha_2 \times k \rceil$	$t+m-\lceil \alpha_2 \times k \rceil+1$	$\ldots$	$t+\lceil \alpha_1 \times k \rceil$	$t+\lceil \alpha_1 \times k \rceil+1$	$\ldots$	$t+m$	$t+m+1$	$\ldots$
Player 1	a_2	a_2	a_2	a_1	a_1	a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_1	a_1
Player 2	b_2	b_2	b_2	b_2	b_2	b_2	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1

The color red denotes actions which necessarily are mistakes. Actions colored blue can be played as a best response.

If $\lceil \alpha_1 \times k \rceil = \lceil \alpha_2 \times k \rceil = m$, then $\ell = m$ and the process described so far has reached convention h_1 with $2 \times m$ mistakes and this convention cannot be reached from h_2 with fewer mistakes. Thus, $r_{2,1}^{k,m} = 2 \times m = \min\big\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big\} + \max\big\{\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil - m, 0\big\}$.

It remains to consider the case when $\lceil \alpha_1 \times k \rceil < m$. In that case, in periods $t+\lceil \alpha_1 \times k \rceil + 1$ through $t+m$, player 2 can sample all $\lceil \alpha_1 \times k \rceil$ of player 1's plays of a_1 in periods $t+1$ through $t+\lceil \alpha_1 \times k \rceil$, and play b_1 as a best response. Because, by definition of ℓ, $m = \big(\lceil \alpha_1 \times k \rceil - \ell\big) + \ell + \big(\lceil \alpha_2 \times k \rceil - \ell\big)$, in these periods, player 1 can sample no more than $\lceil \alpha_2 \times k \rceil - 1$ plays of b_1 and can only play a_2 as a best response.

In period $t+m+1$, it is possible for player 1 to sample all $\lceil \alpha_2 \times k \rceil$ player 2's plays of b_1 in periods $t+m-\lceil \alpha_2 \times k \rceil + 1$ through $t+m$, while player 2 samples all $\lceil \alpha_1 \times k \rceil$ player 1's plays of a_1 in periods $t+1$ through $t+\lceil \alpha_1 \times k \rceil$. Thus, both a_1 and b_1 can be played as best responses by the players in period $t+m+1$ and the process can reach convention h_1 without further mistakes (see Lemma 1).

The process we just described reaches convention h_1 from convention h_2 with exactly $\lceil \alpha_1 \times k \rceil + \ell$ mistakes if $\alpha_1 \le \alpha_2$. Analogously, we can describe a process that reaches convention h_1 from convention h_2 with exactly $\lceil \alpha_2 \times k \rceil + \ell$ mistakes if $\alpha_2 \le \alpha_1$. Thus, we have identified a process that reaches convention h_1 from convention h_2 with exactly the minimum number of mistakes $\min\big\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big\} + \ell$ that we identified as necessary, and we have demonstrated that $r_{2,1}^{k,m} = \min\big\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big\} + \max\big\{\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil - m, 0\big\}$.

Conclusion. We demonstrated that $r_{2,1}^{k,m} = \min\big\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\big\} + \max\big\{\lceil \alpha_1 \times k \rceil + \lceil \alpha_2 \times k \rceil - m, 0\big\}$. The resistance $r_{1,2}^{k,m}$ is now easily obtained by using $1-\alpha_1$ and $1-\alpha_2$ instead of α_1 and α_2, resulting in $r_{1,2}^{k,m} = \min\big\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\big\} + \max\big\{\lceil (1-\alpha_1) \times k \rceil + \lceil (1-\alpha_2) \times k \rceil - m, 0\big\}$. ■

4.2 Stochastically stable conventions

The interest in resistances of moving between conventions stems from the fact that, when the probability of making mistakes (i.e., the degree of perturbation in the process) becomes vanishingly small, the perturbed adaptive play process converges on the conventions that are hardest to leave and easiest to reach. Such conventions are called stochastically stable. In a game with exactly two Nash equilibria, and thus two conventions, a convention is stochastically stable if and only if the resistance in the transition away from it is at least as large as the resistance in the transition towards it (see Young [1993]). In the model studied in Young [1993], sampling is sufficiently incomplete ($k \leq m/2$) for the resistances between conventions to be independent of m, so that stochastic stability of conventions is also independent of m. However, as we demonstrated in Theorem 3, the resistances may be larger and depend on m when sampling is less incomplete ($k > m/2$). This opens up the possibility that the degree of incomplete sampling influences which conventions are stochastically stable. We turn to this next.

It is known that stochastic stability may change with k due to the ceiling functions even when $k \leq m/2$, and the added $\max\{\cdot\}$ component of the resistances can further influence which states are stochastically stable when $k > m/2$. Thus, changing the sample size k when memory size m is fixed may change which states are stochastically stable. However, we demonstrate in Theorem 4 that stochastic stability of conventions does not change when memory size m is changed while keeping sample size k fixed, as long as $m > k$. Thus, while stochastic stability of conventions can be influenced by the sample size, it is not influenced by the degree to which sampling is incomplete.

Theorem 4. *Let G be a 2×2 coordination game. Consider unperturbed adaptive play with fixed sample size k and memory size m such that sampling is incomplete: $m > k$. Stochastic stability of conventions does not depend on memory size m.*

Proof of Theorem 4. We distinguish the two exhaustive cases $\min\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\} \neq \min\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\}$ and $\min\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\} = \min\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\}$. We show that when k is held fixed, changing m does not affect the comparison between the two resistances $r_{1,2}^{k,m}$ and $r_{2,1}^{k,m}$ as long as $m > k$ is maintained.

Case 1. $\min\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\} \neq \min\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\}$.

Without loss of generality assume

$$\min\{\lceil \alpha_1 \times k \rceil, \lceil \alpha_2 \times k \rceil\} < \min\{\lceil (1-\alpha_1) \times k \rceil, \lceil (1-\alpha_2) \times k \rceil\} \tag{1}$$

Then $r_{2,1}^{k,m} < r_{1,2}^{k,m}$ for all $m \geq 2k$. We will show that the inequality $r_{2,1}^{k,m} < r_{1,2}^{k,m}$ is maintained when $m < 2k$, which demonstrates that the comparison between the two resistances is the same for all $m \geq k$.[14]

Using symmetry between players 1 and 2, it follows that we only need to consider one of the cases $\alpha_1 \geq \alpha_2$

[14] The restriction $m > k$ is not leveraged in this case.

and $\alpha_1 \leq \alpha_2$. So, let $\alpha_1 \geq \alpha_2$. Then $\min\{\lceil \alpha_1 \times k\rceil, \lceil \alpha_2 \times k\rceil\} = \lceil \alpha_2 \times k\rceil$ and $\min\{\lceil (1-\alpha_1) \times k\rceil, \lceil (1-\alpha_2) \times k\rceil\} = \lceil (1-\alpha_1) \times k\rceil$, so that (1) translates into $\lceil \alpha_2 \times k\rceil < \lceil (1-\alpha_1) \times k\rceil$. It follows that $\alpha_2 < 1-\alpha_1$ and thus $\lceil \alpha_1 \times k\rceil \leq \lceil (1-\alpha_2) \times k\rceil$, and subsequently

$$\lceil \alpha_1 \times k\rceil + \lceil \alpha_2 \times k\rceil < \lceil (1-\alpha_1) \times k\rceil + \lceil (1-\alpha_2) \times k\rceil \tag{2}$$

Thus, using (1), we obtain $r^{k,m}_{2,1} = \min\{\lceil \alpha_1 \times k\rceil, \lceil \alpha_2 \times k\rceil\} + \max\{\lceil \alpha_1 \times k\rceil + \lceil \alpha_2 \times k\rceil - m, 0\} < \min\{\lceil (1-\alpha_1) \times k\rceil, \lceil (1-\alpha_2) \times k\rceil\} + \max\{\lceil (1-\alpha_1) \times k\rceil + \lceil (1-\alpha_2) \times k\rceil - m, 0\} = r^{k,m}_{1,2}$.

Case 2. $\min\{\lceil \alpha_1 \times k\rceil, \lceil \alpha_2 \times k\rceil\} = \min\{\lceil (1-\alpha_1) \times k\rceil, \lceil (1-\alpha_2) \times k\rceil\}$.

Then $r^{k,m}_{1,2} = r^{k,m}_{2,1}$ for all $m \geq 2k$. We will show that the equality $r^{k,m}_{2,1} = r^{k,m}_{1,2}$ is maintained when $k < m < 2k$.

Using symmetry between players 1 and 2, it follows that we only need to consider one of the cases $\alpha_1 \geq \alpha_2$ and $\alpha_1 \leq \alpha_2$. So, let $\alpha_1 \geq \alpha_2$. Then $\lceil \alpha_2 \times k\rceil = \min\{\lceil \alpha_1 \times k\rceil, \lceil \alpha_2 \times k\rceil\} = \min\{\lceil (1-\alpha_1) \times k\rceil, \lceil (1-\alpha_2) \times k\rceil\} = \lceil (1-\alpha_1) \times k\rceil$.

Using $\lceil \alpha_i \times k\rceil + \lceil (1-\alpha_i) \times k\rceil \in \{k, k+1\}$, $i = 1, 2$, we derive $\lceil \alpha_1 \times k\rceil + \lceil \alpha_2 \times k\rceil = \lceil \alpha_1 \times k\rceil + \lceil (1-\alpha_1) \times k\rceil \leq k+1$ and $\lceil (1-\alpha_1) \times k\rceil + \lceil (1-\alpha_2) \times k\rceil = \lceil \alpha_2 \times k\rceil + \lceil (1-\alpha_2) \times k\rceil \leq k+1$. Because both m and k are natural numbers and $k < m$ it follows that $k+1 \leq m$ and thus $\max\{\lceil \alpha_1 \times k\rceil + \lceil \alpha_2 \times k\rceil - m, 0\} = 0 = \max\{\lceil (1-\alpha_1) \times k\rceil + \lceil (1-\alpha_2) \times k\rceil - m, 0\}$.

We can now aggregate and conclude $r^{k,m}_{2,1} = \min\{\lceil \alpha_1 \times k\rceil, \lceil \alpha_2 \times k\rceil\} + \max\{\lceil \alpha_1 \times k\rceil + \lceil \alpha_2 \times k\rceil - m, 0\} = \min\{\lceil (1-\alpha_1) \times k\rceil, \lceil (1-\alpha_2) \times k\rceil\} + \max\{\lceil (1-\alpha_1) \times k\rceil + \lceil (1-\alpha_2) \times k\rceil - m, 0\} = r^{k,m}_{1,2}$.

Conclusion. We have demonstrated that when k is held fixed, changing m does not affect the comparison between the two resistances $r^{k,m}_{1,2}$ and $r^{k,m}_{2,1}$ as long as $m > k$ is maintained. Because stochastic stability of conventions is determined by the comparisons between these two resistances, it follows that stochastic stability of conventions does not depend on memory size m. ■

Examination of the proof of Theorem 4 reveals that when only one of the conventions is stochastically stable for large m ($m \geq 2k$), we do not need any incomplete sampling to obtain the result that stochastic stability of conventions does not depend on memory size m. The following example shows that when both conventions h_1 and h_2 are stochastically stable for large m, decreasing memory size to $m = k$ may render one of the conventions no longer stochastically stable.

Example 1. *Consider the* 2×2 *coordination game*

Player 2

		b_1	b_2
Player 1	a_1	$10, 11$	$0, 0$
	a_2	$0, 1$	$10, 10$

Suppose that $k = 10$ *and consider the resistances between* h_1 *and* h_2*. Note that in this game* $\alpha_1 = 9/20$ *and* $\alpha_2 = 1/2$*. We compute* $\lceil \alpha_1 \times k \rceil = \lceil \alpha_2 \times k \rceil = \lceil (1 - \alpha_2) \times k \rceil = 5$ *and* $\lceil (1 - \alpha_1) \times k \rceil = 6$*.*

When $m > k = 10$*, the resistances between* h_1 *and* h_2 *are* $r_{1,2}^{k,m} = r_{2,1}^{k,m} = 5$*, and both conventions* h_1 *and* h_2 *are stochastically stable.*

However, when $m = k = 10$*, the resistance of moving from* h_1 *to* h_2 *increases to* $r_{1,2}^{k,m} = 6$ *while the resistance of moving from* h_2 *to* h_1 *remains unchanged at* $r_{2,1}^{k,m} = 5$*. Thus, only convention* h_1 *is stochastically stable.*

Theorem 4 and Example 1 show that making sampling more complete by decreasing memory size can only affect the stochastic stability of conventions in the extreme case that memory size and sample size are equal, and only when the sample size is such that both conventions are stochastically stable under incomplete sampling. In that case, it is possible that one of the conventions is no longer stochastically stable when all records in memory are sampled. We conclude that in 2×2 coordination games, the only bound on the incompleteness of sampling necessary to determine which conventions are stochastically stable is that sampling is incomplete at all, i.e. $k < m$.

5 Conclusion

Young's model of adaptive play and the stochastically stability of conventions have been studied and applied to a wide variety of games. However, the question of precisely what degree of incomplete sampling is necessary to guarantee convergence to a convention has remained unaddressed. We examined this issue in 2×2 coordination games with at least one strict Nash equilibrium. We proved that for all these games the unperturbed adaptive play process converges to a convention for *any* degree of incomplete sampling, and that for almost all games complete sampling also guarantees convergence to a convention provided that sample sizes are large enough.

We then turned to the question whether the degree of incompleteness of the sampling influences which conventions are stochastically stable when the adaptive play process is perturbed. We derived that sample sizes larger than half of the records in memory may result in larger resistances between conventions than smaller sample sizes. However, increasing the sampled proportion of records in memory by decreasing memory size while keeping sample sizes fixed, does not change which conventions are stochastically stable as long as memory size remains strictly larger than the sample size so that sampling is not complete. When memory size is decreased to equal the sample size, and sampling is thus complete, it is possible that one of the conventions is rendered not stochastically stable for some games in which both conventions are stochastically stable when sampling is incomplete.

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